On the determination of non-local symmetries

1995 J. Phys. A: Math. Gen. 28 5349

(http://iopscience.iop.org/0305-4470/28/18/023)

J. Phys. A: Math. Gen. 28 (1995) 5349–5359. Printed in the UK

On the determination of non-local symmetries

K S Govinder†§ and P G L Leach‡§‖

Department of Mathematics, University of the Aegean, Karlovassi 83200, Samos, Greece

Received 17 November 1994

Abstract. The importance of non-local symmetries of differential equations lies in their manifestation as Lie point symmetries of the equations resulting from reduction of order. The reason for the determination of these symmetries in second-order equations with only one Lie point symmetry is self-evident. However, the disadvantage of non-local symmetries is that no systematic approach to their determination exists. We present such an approach (applicable to differential equations of any order) and apply it to some second-order ordinary differential equations and show that they have a rich occurrence. We also look at possible generalizations of the concept of non-local symmetries.

1. Introduction

The concept of invariance under transformation is central to understanding the mathematical description of physical phenomena and the solution of equations which comprise that description. Thus the conservation of energy is associated with invariance of the Hamiltonian under time translation and that of angular momentum with invariance under rotation. Various theories have been developed to exploit the utility of conservation laws in the solution of physical problems. To take a typical example, a Hamiltonian, $H(q, p, t)$, is said to have a first integral, $I(q, p, t)$, if I is a solution of Liouville's equation:

$$\dot{I} = \frac{\partial I}{\partial t} + [I, H]_{\mathrm{Pb}} = 0 \tag{1.1}$$

where $[\ ]_{\mathrm{Pb}}$ is the usual Poisson bracket. For an autonomous Hamiltonian, $H(q, p)$, (1.1) has the obvious solution

$$I(q, p) = H(q, p). \tag{1.2}$$

This may be of some comfort, but it does not advance the solution of the Hamiltonian system. It is also not possible to progress further in the solution of Lagrange's systems associated with (1.1), viz

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}q}{\partial H/\partial p} = \frac{\mathrm{d}p}{-\partial H/\partial q} \tag{1.3}$$

† E-mail: kgov@athena.aegean.ariadne-t.gr

‡ E-mail: leach@athena.aegean.ariadne-t.gr

§ Permanent address: Department of Mathematics and Applied Mathematics, University of Natal, Private Bag X10, Dalbridge 4014, South Africa (E-mail: govinder@ph.und.ac.za, leach@ph.und.ac.za).

‖ Member of the Centre for Theoretical and Computational Chemistry, University of Natal, Durban, South Africa, and associate member of the Centre for Nonlinear Studies, University of the Witwatersrand, Johannesburg, South Africa

0305-4470/95/185349+11$19.50

for general $H(q, p)$, let alone $H(q, p, t)$, unless there is some information about the internal structure of H, for example, that it is of 'natural' form

$$H = \tfrac{1}{2}p^2 + V(q). \tag{1.4}$$

Even this is not sufficient if $V = V(q, t)$ and the time dependence is intricately involved in the potential.

What must be done then is that a structure be assumed for I, typically by specifying the nature of its dependence on the momentum, p. Polynomials [1, 2] and rational functions [3, 4] are the forms usually assumed. If there exists a first integral and it is of the form assumed, all is well and good apart from perhaps a certain number of technicalities interposed between ansatz and solution. The absence of a solution does not mean that a first integral does not exist, only that one has not been sufficiently perspicacious to guess its form.

Clearly the least possible restraints placed on the form assumed for the first integral is the optimal route to obtaining the most general results. Systematic approaches to the use of invariance under transformation are found in the use of symmetries as generators of infinitesimal transformation in Noether's theorem [5] and the Lie theory of extended groups. The degree of generality of the results to be potentially obtained by the Lie method depends upon the ansatz made about the functional dependence of the functions, ξ and η, which describe the generator of symmetry, G, as

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{1.5}$$

(add indices for more than one dependent variable). The standard approach is to have ξ and η as functions of x and y only. These point symmetries [6, p 47] can be straightforwardly calculated for any differential equation of order higher than first order and any function higher than zeroth order. If ξ and η are permitted to depend on derivatives, their calculation can become problematic except in the case of contact symmetries of equations of higher order than the second.

In a number of papers Abraham-Shrauner and co-workers [7–12] have discussed what they term hidden symmetries. There are two varieties. Hidden symmetries of Type I arise when the order of an equation is increased and Type II when the order of an equation is decreased. As our primary concern is the reduction of the order of equations, hidden symmetries of Type II are the ones of interest to us. (See [13] for applications of Type-I hidden symmetries.) The origin of Type-II hidden symmetries is found in non-local symmetries of the higher-order equation. It is appropriate that we make precise the different varieties of symmetry before continuing and their common meaning. Each of them, when appropriately extended to be able to act on all derivatives present, gives zero when acting on the differential equation for which it is a symmetry. The symmetry is point if the coefficient functions ξ and η depend upon the dependent and independent variables only, generalized (a subset being contact) if they depend upon derivatives as well and non-local if they depend upon an integral or, indeed, multiple integrals.

For the purposes of reduction of order we are interested in non-local symmetries which become point for the equation of reduced order. This type of non-local symmetry we call first order and we have

$$G = \xi(x, y, I)\frac{\partial}{\partial x} + \eta(x, y, I)\frac{\partial}{\partial y} \tag{1.6}$$

where

$$I = \int f(x, y)\, dx. \tag{1.7}$$

We shall see in section 3 that even (1.7) is too general, but it shall suffice for the present.

In the following sections we investigate the implications of allowing the generalization to non-local symmetries on the reduction of differential equations to quadratures.

2. Determination of non-local symmetries

We recall that an nth-order differential equation

$$E(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{2.1}$$

(We will only consider nth-order scalar ordinary differential equations in the ensuing discussion. The results can, in principle, be extended to systems and partial differential equations. However, in practice this process may be decidedly non-trivial [14].) has the Lie point symmetry

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.2}$$

if

$$X^{[n]}E_{|_{E=0}} = 0 \tag{2.3}$$

where [15]

$$X^{[n]} = X + \sum_{i=1}^{n}\left[\eta^{(i)} - \sum_{j=0}^{i-1}\begin{pmatrix} i \\ j \end{pmatrix} y^{(j+1)}\xi^{(i-j)}\right]\frac{\partial}{\partial y^{(i)}}. \tag{2.4}$$

We call $X^{[n]}$ the nth extension (or prolongation in the terminology of [16]) which is needed to contend with the nth and lower derivatives in (2.1). Note that ξ and η in (2.2) depend on x and y only. This restriction yields Lie point symmetries only which are normally used to reduce (2.1) to quadratures. Note that X is the generator of the group of infinitesimal point transformations

$$\bar{x} = x + \varepsilon\xi \tag{2.5a}$$

$$\bar{y} = y + \varepsilon\eta \tag{2.5b}$$

that leave (2.1) invariant. In this paper we broaden this class of transformations to non-local transformations.

For the purposes of the ensuing discussion we call the set of infinitesimal transformations

$$\bar{x} = x + \varepsilon\xi \tag{2.6a}$$

$$\bar{y} = y + \varepsilon\eta \tag{2.6b}$$

$$\bar{I} = I + \varepsilon\gamma \tag{2.6c}$$

where

$$I = \int f(x, y)\,\mathrm{d}x \tag{2.7}$$

a first-order one-parameter Lie group of non-local infinitesimal transformations. The generator of these non-local transformations is

$$G = \xi(x, y, I)\frac{\partial}{\partial x} + \gamma(x, y, I)\frac{\partial}{\partial I} + \eta(x, y, I)\frac{\partial}{\partial y} \tag{2.8}$$

where

$$\frac{\partial\gamma}{\partial y} - y\frac{\partial\xi}{\partial y} = 0. \tag{2.9}$$

We require (2.9) to remove the possibility of derivatives in η and thereby that the space of transformations closes. The above notation is given to show the link between our concept of non-local symmetries and that of the classic Lie point symmetries.

Remark. In general we could require ξ, γ and η in (2.8) to depend on x, y, y' and $I = \int f(x, y, y')\,\mathrm{d}x$. We address this possibility in section 4.

Note that we do not need to include η in (2.6) and (2.8) as the first extension of

$$\tilde{G} = \xi(x, y, I)\frac{\partial}{\partial x} + \gamma(x, y, I)\frac{\partial}{\partial I} \tag{2.10}$$

defines η as

$$\eta = \frac{\mathrm{d}\gamma}{\mathrm{d}x} - y\frac{\mathrm{d}\xi}{\mathrm{d}x}. \tag{2.11}$$

However, in practice one knows ξ and η and works backwards to determine γ. In the actual calculation of non-local symmetries we ignore $\gamma\partial/\partial I$ in (2.8) as we are only concerned with differential equations. We remark that the generator defined in (2.8) has a 'contact-symmetry-like' structure.

We are now in a position to calculate the non-local symmetries of (2.1). The procedure is similar to that of determining its Lie point symmetries. We require (2.1) to be invariant under *the nth extension of (2.8). The main difference between the resulting calculation and* that for point symmetries is the introduction of $\partial/\partial I$ terms. The determining equations form a system of linear *ordinary* differential equations.

3. Non-local symmetries of second-order ordinary differential equations

While it is of mathematical interest to determine non-local symmetries of differential equations in general, the important occurrence of these symmetries is in second-order ordinary differential equations. Remember that non-local symmetries of a differential equation manifest themselves as Type-II hidden symmetries of the reduced equation. Thus a simple reduction of order and subsequent calculation of the Lie point symmetries (e.g. using *Program LIE* [17]) will determine the 'useful' non-local symmetries of any equation. (We define 'useful' non-local symmetries as those that reduce to point symmetries under a single reduction of order of the equation.) While this is true in general, it does not apply for second-order equations as there is no direct method to determine the infinite number of point symmetries that arise in the reduced first order equation [18]. In the instance that the second-order equation possesses more than one point symmetry, reduction of order via the appropriate point symmetry (i.e. one which does not annihilate the others as point symmetries [16, p 149]) will result in a first-order equation with at least one known point symmetry. Thus the case of second-order equations possessing just one point symmetry is the one of paramount importance. The determination of at least one 'useful' non-local symmetry for such equations provides a systematic route to the solution of such equations via the classical Lie theory of extended groups [16, 18].

We analyse the equation

$$E(y, y', y'') = y'' - g(y, y') = 0 \tag{3.1}$$

with the sole Lie point symmetry

$$G_1 = \frac{\partial}{\partial x} \tag{3.2}$$

for the existence of non-local symmetries. The restriction to (3.1) causes no loss of generality, as all ordinary differential equations (not just those of second order) with at least one symmetry can always be transformed to autonomous form.

In the case of (3.1) possessing two point symmetries the Lie bracket relationship

$$[G_1, G_2]_{\mathrm{Lb}} = \lambda G_1 \tag{3.3}$$

where λ is a constant (either 0 or scaled to 1), guarantees G_2 as a point symmetry of the reduced equation. If G_1 is defined as in (3.2) and

$$G_2 = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{3.4}$$

equation (3.3) implies that G_2 must have the form

$$G_2 = (\lambda x + k(y))\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{3.5}$$

The reduction of (3.1) by the transformation generated by (3.2), viz

$$u = y \qquad v = y' \tag{3.6}$$

results in the first-order equation

$$vv' = g(u, v). \tag{3.7}$$

Thus G_2 will reduce (using equation (3.6) and the first extension of (3.5)) to

$$Y = a(u)\frac{\partial}{\partial u} + v(a'(u) - \lambda - k'(u)v)\frac{\partial}{\partial v}. \tag{3.8}$$

(See also [18, p 129].)

The structure of a non-local symmetry will be, in general, (disregarding the $\partial/\partial I$ term)

$$G_{\mathrm{nl}} = \xi(x, y, I)\frac{\partial}{\partial x} + \eta(x, y, I)\frac{\partial}{\partial y} \tag{3.9}$$

where

$$I = \int f(x, y)\,\mathrm{d}x. \tag{3.10}$$

However, by noting the form of (3.8) we deduce that

$$\eta = a(y) \qquad \text{and} \qquad \xi = \lambda x + k(y, I). \tag{3.11}$$

This follows from the requirement

$$[G_1, G_{\mathrm{nl}}] = \lambda G_1. \tag{3.12}$$

The non-local symmetry of (3.1) now reduces to

$$G_{\mathrm{nl}} = (\lambda x + k(y, I))\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{3.13}$$

The requirement that ξ be free of y' and ξ' be free of x and I gives

$$G_{\mathrm{nl}} = (\lambda x + I)\frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y} \tag{3.14}$$

where

$$I = \int c(y)\,\mathrm{d}x. \tag{3.15}$$

This form is more restrictive than just being of first order (i.e. more restrictive than (1.6), (1.7)). Higher-order non-local symmetries (with multiple integrals) cannot be useful in the sense defined above as they will not reduce to point symmetries under a single reduction of order.

To determine the coefficient functions in (3.14) we require

$$G_{\rm nl}^{[2]}E|_{E=0} = 0 \tag{3.16}$$

i.e.

$$-a\frac{\partial g}{\partial y} + ((\lambda + c)y' - y'a')\frac{\partial g}{\partial y'} + ((a' - 2(\lambda + c))g + y'^2(a'' - c')) = 0. \tag{3.17}$$

The solution of the associated Lagrange's system reduces to that of two linear first-order ordinary differential equations which we solve [19] for g to obtain

$$g = \mathrm{e}^{-\int \phi\, \mathrm{d}y}\left[\int \psi \mathrm{e}^{\int \phi\, \mathrm{d}y}\mathrm{d}y + L(u)\right] \tag{3.18}$$

where

$$\begin{aligned}\phi &= \frac{a' - 2(\lambda + c)}{a}\\ \psi &= \frac{a(c' - a'')}{u^2}\exp\left(-2\int \frac{\lambda + c}{a}\,\mathrm{d}y\right)\\ u &= \frac{a}{y'}\exp\left(-\int \frac{\lambda + c}{a}\,\mathrm{d}y\right).\end{aligned} \tag{3.19}$$

To make this implicit result clearer it is useful to look at a practical example.

Consider the equation [21]

$$R'' + \frac{\varepsilon R^3}{(\rho^2 + 1)^3} - \frac{K^2}{R^3} = 0 \tag{3.20}$$

where K is a constant and ε a parameter, which is a reduced form of the complex Lorenz system under certain assumptions about its parameters. Equation (3.20) has the single Lie point symmetry

$$G = (\rho^2 + 1)\frac{\partial}{\partial \rho} + \rho R\frac{\partial}{\partial R}. \tag{3.21}$$

We use (3.21) to rewrite (3.20) in autonomous form, viz

$$y'' + \varepsilon y^3 + y - \frac{K^2}{y^3} = 0 \tag{3.22}$$

via the transformation

$$x = \tan^{-1}\rho \qquad y = R(\rho^2 + 1)^{-1/2}. \tag{3.23}$$

We can now analyse (3.22) for non-local symmetries of the form (3.14). We find that

$$\begin{aligned}a &= \frac{1}{p}\left(A_1 + A_0\int p\,\mathrm{d}y\right)\\ c &= -\frac{p'}{p^2}\left(A_1 + A_0\int p\,\mathrm{d}y\right) + \bar{A}_0\\ p &= \varepsilon y^3 + y - \frac{K^2}{y^3} \qquad \bar{A}_0 = \tfrac{1}{2}A_0 - \lambda\end{aligned} \tag{3.24}$$

where A_0 and A_1 are constants and the $'$s on $p(y)$ mean differentiation with respect to its argument, y.

It is significant that (3.22) has at least two Lie symmetries of the form (3.14) that commute. This guarantees the reduction of (3.20) to quadratures. It has already been shown that (3.20) possesses the Painlevé property [22] and is hence conjectured to be integrable [23]. (See [24] for a good exposition of the Painlevé property of differential equations.) The reduction to quadrature has already been performed [21] without knowledge of the non-local symmetry. The occurrence of this symmetry is further evidence of the close relationship between the Lie and Painlevé analyses of differential equations as has been indicated previously [25, 26].

4. Generalized non-local symmetries

Thus far we have only considered 'point-like' non-local symmetries, i.e. the dependence of the coefficient functions was free from derivatives. This was due to the fact that we were working from a knowledge of point symmetries. However, there is no reason to exclude more general non-local symmetries, as they can reduce to point symmetries under a single reduction of order and are also 'useful'.

Consider the generator of non-local infinitesimal transformations

$$Z_{\mathrm{nl}} = \xi\left(x, y, y', \int f(x, y, y')\,\mathrm{d}x\right)\frac{\partial}{\partial x} + \eta\left(x, y, y', \int f(x, y, y')\,\mathrm{d}x\right)\frac{\partial}{\partial y}. \qquad (4.1)$$

For (4.1) to be a 'useful' non-local symmetry it has to reduce to

$$G_{\mathrm{red}} = a(u, v)\frac{\partial}{\partial u} + b(u, v)\frac{\partial}{\partial v} \qquad (4.2)$$

under

$$u = y \qquad v = y'. \qquad (4.3)$$

This restriction, together with the analogue of (3.12), confines the analysis to generalized non-local symmetries of the form

$$Z_{\mathrm{nl}} = \int c(y, y')\,\mathrm{d}x\frac{\partial}{\partial x} + a(y, y')\frac{\partial}{\partial y}. \qquad (4.4)$$

(Here we have taken $\lambda = 0$ for simplicity.) With c and a being arbitrary functions of y and y' the analysis can only proceed to writing down the equation to be solved. (The problem is similar to calculating contact symmetries of second-order ordinary differential equations.) Further progress can only be made by assuming *a priori* a form for the non-local symmetry (4.4). We look at some special cases.

4.1. $a(y, y') = 0$

We require

$$E(y, y', y'') = y'' - g(y, y') = 0 \qquad (4.5)$$

to be invariant under (4.4) with $a(y, y') = 0$. This results in g having the following form

$$g = \frac{1}{cy'^2}\left[F(y) - y^2\frac{\partial}{\partial y}\left(\int cy'\,\mathrm{d}y'\right)\right] \qquad (4.6)$$

for (4.5) to have a generalized non-local symmetry. Given g we can determine c via (4.6) (Note that $F(y)$ is an arbitrary function of y.) to obtain (4.4) with $a(y, y') = 0$.

4.2. $a(y, y') = 0$, $g = g(y)$

In this example we require

$$E(y, y'') = y'' - g(y) = 0 \tag{4.7}$$

to be invariant under

$$Z_{\text{nl}} = \int c(y, y')\, \mathrm{d}x \frac{\partial}{\partial x}. \tag{4.8}$$

For a given g, c has the form

$$c = \frac{F\left(\frac{1}{2}y'^2 - \int g(y)\, \mathrm{d}y\right)}{y'^2} \tag{4.9}$$

where F is an arbitrary function of its argument. Note that the Lorentz system (3.22) falls into the class (4.7). This implies that (3.22) has an additional non-local symmetry of the form (4.8). The occurrence of the additional symmetry is unsurprising as the first-order equation obtained from (3.22) under the reduction (4.3) will have an infinite number of point symmetries. We expect to find, in principle, an infinite number of 'useful' non-local symmetries of (3.22). The form of these symmetries will depend on our ansätze for $a(y, y')$ and $c(y, y')$.

The class of equations considered by Guo and Abraham-Shrauner [12] is also contained in (4.7). They found that (4.7) was invariant under

$$Z_{\text{nl}} = \int \frac{1}{y'^2}\, \mathrm{d}x \frac{\partial}{\partial x} \tag{4.10}$$

by first considering the first-order equation that results from the reduction of (4.7) via (4.3). Our method has resulted in a generalization of their results.

4.3. $a = a(y)$, $c = c(y')$, $g = g(y)$

We look at

$$y'' = g(y) \tag{4.11}$$

again. This time we require it to be invariant under

$$Z_{\text{nl}} = \int c(y')\, \mathrm{d}x \frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial y}. \tag{4.12}$$

For a given g, a and c are related via

$$g = A_0 a \exp\left[-\left(2c + y'\frac{\mathrm{d}c}{\mathrm{d}y'}\right)\int \frac{1}{a}\mathrm{d}y\right] + y'^2 \int \left[\frac{1}{a^2}\frac{\mathrm{d}^2 a}{\mathrm{d}y^2}\exp\left(-\int \frac{1}{a}\, \mathrm{d}y\right)\, \mathrm{d}y\right] \tag{4.13}$$

where A_0 is an arbitrary constant of integration.

4.4. *Equations linear in* y'

Equations that are at most linear in y', e.g.

$$y'' = \alpha y' + g(y) \tag{4.14}$$

where α is a constant, are of some interest as they reduce to an Abel equation [20] under (4.3). Unfortunately, there is no simple way to compute its non-local symmetry, e.g. non-local symmetries of the forms

$$Z_{\text{nl}} = \int c(y')\, \mathrm{d}x \frac{\partial}{\partial x} \tag{4.15}$$

and

$$\tilde{Z}_{\rm nl} = \int c(y)\, \mathrm{d}x \frac{\partial}{\partial x} \tag{4.16}$$

do not leave (4.14) invariant. We obviously need to make a more complicated ansatz. It is whimsical to note that the choice $c = c(y, y')$ and $a = 0$ reduces the problem to that of solving an Abel equation to determine the non-local symmetry! See also [27] for a discussion of methods to obtain solutions for Abel's equation and the impact of hidden symmetries.

Similar observations should aid in the choice of ansätze for a and c in (4.4). It is apparent that these symmetries are most appropriate to third-order equations.

5. Conclusion

We have presented a systematic approach for finding first-order and generalized non-local symmetries of second-order ordinary differential equations. While the solutions of the resulting equations may look complicated their determination is surprisingly straightforward (we need to solve linear first-order ordinary differential equations as opposed to the linear partial differential equations of the classical method.). The analysis was confined to the determination of those non-local symmetries that reduce to point symmetries under reduction of order via

$$G = \frac{\partial}{\partial x} \tag{5.1}$$

where x represents the independent variable. This restriction is valid as we are only interested in those non-local symmetries of the second-order equation that allow us to reduce the resulting first-order equation to quadratures.

Unfortunately, due to the arbitrary nature of the coefficient functions we often need to impose further restrictions on the structure of the non-local symmetry. Usually (in the case of point symmetries) this arbitrariness does not present a difficulty. Here the problem lies in the arbitrary nature of the integrand in these functions. However, we have still managed to make some progress. (The restrictions imposed on the integrand are justified by the requirement that the non-local symmetry commutes with (5.1) and that it becomes a point symmetry under the reduction of order. These assumptions are necessary for the non-local symmetry to be of practical use.) In particular we have been able to classify all second-order equations possessing a first-order non-local symmetry in addition to the point symmetry (5.1). The further classification of second-order equations using non-local symmetries lies in making an appropriate ansatz for the integrand in the coefficient functions. A few examples (not meant to be exhaustive) were given to illustrate the principle.

We remarked earlier that the search for non-local symmetries should be confined to second-order equations. In the case that one is dealing with higher-order systems where the reductions are non-trivial it may be of some benefit to analyse those systems for first- and higher-order non-local symmetries. We leave it to the practitioner to decided which of the two approaches is optimal.

We note that differential equations have a rich structure of non-local symmetries. This was amply illustrated by the reduced form of the complex Lorentz system. A simple example is the analysis of

$$y'' = 0 \tag{5.2}$$

for non-local symmetries of the form

$$G_{\mathrm{nl}} = \xi\left(x, y, \int y\,\mathrm{d}x\right)\frac{\partial}{\partial x} + \eta\left(x, y, \int y\,\mathrm{d}x\right)\frac{\partial}{\partial y}. \tag{5.3}$$

Even with this severe restriction on the integrand we find a large number of non-local symmetries.

We have only concentrated on linear non-local symmetries in this paper. A method for the systematic search for other possible non-local symmetries (e.g. involving exponential functions [8] or other elementary functions) would be of some interest. The determination of exponential non-local symmetries are of particular interest as they allow us to reduce the order of equations that do not necessarily possess any point symmetries [8] and are therefore different from the examples considered here.

A final remark, in order to place this work in its proper perspective, is that hidden symmetries have recently been explained from a geometric viewpoint [28] using the concept of solvable structures [29]. It is hoped that the treatment above complements this approach.

Acknowledgments

We thank B Abraham-Shrauner, F M Mahomed and N H Ibragimov for their kind and constructive comments and the University of Natal and the Foundation for Research Development of South Africa for their ongoing support.

References

[1] Gascón F G, Ramas F B and Aguirre-Dabon E 1982 *J. Math. Phys.* **23** 2281
[2] Lewis H R and Leach P G L 1982 *J. Math. Phys.* **23** 2371
[3] Lewis H R and Leach P G L 1985 *Ann. Phys., NY* **164** 47
[4] Taylor D R and Leach P G L 1990 *Ann. Phys., NY* **198** 358
[5] Sarlet W and Cantrijn F 1981 *SIAM Review* **23** 467
[6] Lie S 1893 *Vorlesungen über Continuierliche Gruppen mit Geometrischen und Anderen Anwendungen (Bearbeitet und Herausgegeben von G Scheffers)* (reprinted 1971 (New York: Chelsea))
[7] Abraham-Shrauner B 1993 *J. Math. Phys.* **34** 4809
[8] Abraham-Shrauner B 1994 Nonlocal group generators *Eur. J Appl. Math.* submitted
[9] Abraham-Shrauner B and Guo A 1992 *J. Phys. A: Math. Gen.* **25** 5597
[10] Abraham-Shrauner B and Guo A 1993 Hidden and nonlocal symmetries of nonlinear differential equations *Modern Group Analysis: Advanced Analytical and Computational methods in Mathematical Physics* ed N H Ibragimov, M Torrissi and A Valenti (Dordrecht: Kluwer) p 1
[11] Abraham-Shrauner B and Leach P G L 1993 Hidden symmetries of nonlinear ordinary differential equations *Exploiting Symmetry in Applied and Numerical Analysis (Lectures in Applied Mathematics 29)* (Providence, RI: American Mathematical Society) p 1
[12] Guo A and Abraham-Shrauner B 1993 *IMA J. Appl. Math.* **51** 147
[13] Govinder K S and Leach P G L 1995 A Group theoretic approach to a class of second order ordinary differential equations not possessing Lie point symmetries *Preprint* Department of Mathematics, University of the Aegean
[14] Mahomed F M 1994 private communication
[15] Mahomed F M and Leach P G L 1990 *J. Math. Anal. Appl.* **151** 80
[16] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[17] Head A K 1993 *Comput. Phys. Commun.* **77** 241
[18] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)
[19] Ince E L 1927 *Ordinary Differential Equations* (London: Longmans, Green and Co)
[20] Kamke E 1983 *Differentialgleichungen Lösungsmethoden und Lösungen* vol I (Stuttgart: Teubner)
[21] Flessas G P and Leach P G L 1991 *Phys. Lett.* **154A** 127
[22] Roekaerts D 1988 *J. Phys. A: Math. Gen.* **21** L495
[23] Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715, 1006

[24] Conte R 1994 Singularities of differential equations and integrability *Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Nonlinear Waves* ed D Benest and C Frœschlé (Gif-sur-Yvette: Éditions Frontières) p 49
[25] Lemmer R L and Leach P G L 1993 *J. Phys. A: Math. Gen.* **26** 5017
[26] Govinder K S, Leach P G L and Maharaj S D 1995 *Int. J. Theor. Phys.* **34** 625
[27] Adam A A, Mahomed F M and Kara A H 1994 *Lie Groups and Their Applications* **1** 1
[28] Hartl T and Athorne C 1994 *J. Phys. A: Math. Gen.* **27** 3463
[29] Basarab-Horwath P 1991 *Ukrainskii Matematicheskii Zhurnal* **43** 1330